Mathematical Statistics Qualifier Examination
Part I of the STAT AREA EXAM
May 29, 2019; 9:00 AM - 11:00 AM

NAME: ______________________________ **ID:** ______________________________

Signature: ____________________________________

Instruction: There are 4 problems. You are required to solve them all. Please show detailed work for full credit. This is a closed book exam from 9 am to 11 am. You need to turn in your exam by 11 am, and subsequently receive the questions for your applied statistics exam. Please do NOT use a calculator or cell phone. Good luck!

1. (a) Let the random variable X possess a folded normal distribution with pdf

$$f(x) = \sqrt{\frac{2}{\pi}} \exp(-x^2/2), \quad 0 < x < \infty.$$

Find the mean and variance of X.

(b) A random variable X possesses a skewed-normal distribution with mode μ and scale parameters σ_1^2 and σ_2^2 if its density is given by

$$f\left(x|\mu, \sigma_1^2, \sigma_2^2\right) = \begin{cases} \left(\frac{2}{\pi}\right)^{1/2} (\sigma_1 + \sigma_2)^{-1} \exp\left\{-\frac{1}{2}\sigma_1^{-2}(x-\mu)^2\right\} & \text{for } -\infty < x \leq \mu \\ \left(\frac{2}{\pi}\right)^{1/2} (\sigma_1 + \sigma_2)^{-1} \exp\left\{-\frac{1}{2}\sigma_2^{-2}(x-\mu)^2\right\} & \text{for } \mu < x < \infty. \end{cases}$$

i. Find $P(X \leq \mu)$.

ii. Find $\mathrm{E}X$.

2. Suppose that X has conditional cdf

$$F(x|\theta) \propto 1 - e^{-\theta \sin x}, \; 0 \le x \le \pi/2; \; \theta > 0.$$

(a) Find the conditional pdf $f(x|\theta)$ of X.

(b) Given $x_1, x_2, \cdots, x_n$, a random sample of observation of X, derive an equation which the MLE $\hat{\theta}$ of θ must satisfy.

(c) Find an approximate expression for the posterior variance of θ for the case where n is large. You may assume that the prior density of θ is non-zero for $\theta > 0$.

3. A sequence of Bernoulli trials with success probability θ $(0 < \theta < 1)$ is performed until the kth failure is observed $(k > 1)$. Find the UMVUE of θ.

4. Let $X_1, \ldots, X_n$ be a random sample from $N(0, \sigma^2)$. Consider a test for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \neq \sigma_0$.

   (a) Formulate the UMPU level α test.

   (b) Let $g_n(u)$ be the pdf of the χ^2_n distribution given as

   $$g_n(u) = \frac{1}{\Gamma(n/2)2^{n/2}} u^{\frac{n}{2}-1} e^{-u/2}, \ u > 0$$

   Show that $u g_n(u)/n = g_{n+2}(u)$.

   (c) Show that the two conditions for the test in part (a) are

   $$\int_{c_1}^{c_2} g_n(u) du = 1 - \alpha \quad \text{and} \quad \int_{c_1}^{c_2} g_{n+2}(u) du = 1 - \alpha.$$